Total Factor Productivity under a Multi-Period and Multi-Agent Model

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This paper attempts to apply a more general setting of the multi-period and multi-agent type model with state contingent actions to the analysis of total factor productivity. When households face different linearly homogenous production functions, an estimated average growth rate of technical change could be different from total factor productivity growth, introduced by Solow (1957). Its implication is that total factor productivity growths of some sectors could be affected by output of a few large enterprises that have huge output shares and high capital intensity. The implication for applied researches is that one could get better estimates of total factor productivity growths for some sectors if one is able to estimate total factor productivity growths of large and small firms separately.

1. Introduction

The concept of total factor productivity proposed by Solow (1957) has been, from the beginning, built on the basis of aggregate production function and competitive markets. This paper attempts to apply a more general setting of the multi-period and multi-agent type model with state contingent actions into the analysis of total factor productivity.

When households face different linearly homogenous production functions, an average growth rate of technical change could be different from Solow's total factor productivity because the estimated average growth of technical change is weighted by output share. Its implication is that total

factor productivity growths of some sectors could be highly sensitive to output of a few large enterprises with huge output shares and high capital intensity.

The implication for applied researches is that one may get better estimates of total factor productivity growths for some sectors if one chooses to estimate total factor productivity growths of large and small firms separately.

2. The model

The formal setting of a hypothetical economy used in this paper comes from a more general setting of a village economy studied by Townsend (1993). In this model, an agricultural economy has multiple dates which are denoted by $t = 0, 1, ..., T$. There are many risk aversive households, denoted by j, $j = 1, 2,...,J$. A risk neutral landlord, denoted by $J+1$, owns all the existing agricultural land. In the case of rent sharing, each household (j) must give part of his output to the landlord for the use of land. Each household (j) has a linearly homogenous production function, $Q_t^j = g_t^j (\eta_t^A) f^j (N_t^j, L_t^j)$,[2] where N_t^j and L_t^j are household (j)'s effort to work and total amount of land, while the function $g_t^j (\eta_t^A)$ appears to allow for household (j)'s experience in technological change as a result of aggregate shocks η_t^A. Labor effort and output can be bought or sold in the markets where prices are given. Let $P_{q,t} (\eta_t^q)$ be the price for crop output at date t as a function of aggregate shocks η_t^q, and $P_{a,t}(\eta_t^a)$ be the price for labor effort as a function of aggregate shocks η_t^a. Land rent is denoted by $P_{r,t}(\eta_t^r)$ and is a function of aggregate shocks η_t^r. It is assumed that both labor and land are homogeneous factors for all households.

A state is characterized by the realizations of aggregate shocks,

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[2] The assumptions of linearly homogeneous production function and neutral technical change are employed to simplify the analysis.

η_t^A, η_t^q, η_t^a and η_t^r, and household specific shocks ε_t^H. Let ε_t denote the entire vector of all shocks. State ε_t completely describes the realizations of all random variables relevant to this economy. The entire contemporary state ε_t and the entire prior history $(\varepsilon_0,....,\varepsilon_{t-1})$ can describe completely each household (j)'s consumption, c_t^j $(\varepsilon_0,....,\varepsilon_t)$, and leisure, ℓ_t^j $(\varepsilon_0,....,\varepsilon_t)$. Let π $(\varepsilon_0,....,\varepsilon_t)$ denote the probability of this entire vector as of some planning date t = 0.

3. The full information economy

This section gives analysis of the two different cases in full information economy. The first is the case where all households have to pay the same market land rent for the use of land. The second case is the case of rent sharing where each household (j) pays part of his output to the landlord for the use of land.

3.1 The case where all households pay the same market land rent

Pareto optima for this economy ensures that each household (j)'s consumption at each date is to co- move with aggregate output. Such a result can be obtained by solving the following problem,

$$
\underset{\ell_t^j, c_t^j}{Max} \sum_{j=1}^J \omega^j \left[\sum_{t=0}^T \beta^t \sum_{(\varepsilon_0...\varepsilon_t)} \pi\left(\varepsilon_0...\varepsilon_t\right) U^j\left[c_t^j(\varepsilon_0...\varepsilon_t), \ell_t^j(\varepsilon_0...\varepsilon_t)\right] \right] \qquad (1)
$$

where $(\omega^1, \omega^2,...,\omega^j)$ captures each household (j)'s relative weight in the economy. Let β^t be the discount factor that discounts any future value of date t into the present value at date t = 0. Utility function of each household (j), j = 1,2,..., J, is denoted by $U^j(\cdot)$ which, for simplicity, satisfies these properties:

$$
\begin{aligned}
&\partial U^j (\cdot)/\partial c_t^j > 0, \partial^2 U^j(\cdot)/\partial c_t^{j\,2} < 0, \\
&\partial U^j (\cdot)/\partial \ell_t^j > 0, \\
&\partial^2 U^j (\cdot)/\partial \ell_t^{j\,2} < 0, U^j (0,0) = 0
\end{aligned} \qquad (2)
$$

Furthermore, each household can buy or sell labor service at the market price. Each household can also rent land from the landlord at the market rent. Consequently, the problem can be written as

$$\underset{\mathcal{L}_t^j, C_t^j, N_t^j, L_t^j}{Max} \sum_{j=1}^{J} \omega^j \left[\sum_{t=0}^{T} \beta^t \sum_{(\varepsilon_0..\varepsilon_t)} \pi\,(\varepsilon_0.,\varepsilon_t\,) U^j \left[c_t^j(\varepsilon_0.,\varepsilon_t\,), \mathcal{L}_t^j(\varepsilon_0.,\varepsilon_t\,) \right] \right] \tag{3}$$

subject to the economy aggregate budget constraint,

$$P_{q,t}(\eta_t^q) \sum_{j=1}^{J} c_t^j(\varepsilon_0.,\varepsilon_t\,) + P_{a,t}(\eta_t^a) \sum_{j=1}^{J} N_t^j(\varepsilon_0.,\varepsilon_t\,)$$

$$+ P_{r,t}(\eta_t^r) \sum_{j=1}^{J} L_t^j(\varepsilon_0.,\varepsilon_t\,)$$

$$= P_{q,t}(\eta_t^q) \sum_{j=1}^{J} g_t^j(\eta_t^A) f^j\left(N_t^j(\varepsilon_0.,\varepsilon_t\,), L_t^j(\varepsilon_0.,\varepsilon_t\,) \right) \tag{4}$$

$$+ P_{a,t}(\eta_t^a) \left[\sum_{j=1}^{J} \left(\bar{\mathcal{L}}_t^j\left(\varepsilon_t^H\right) - \mathcal{L}_t^j(\varepsilon_0.,\varepsilon_t\,) \right) \right],$$

$$t = 0,1,...,T\;;\; j = 1,2,...,\,J.$$

where $\bar{\mathcal{L}}_t^j\left(\varepsilon_t^H\right)$ is household (j)'s leisure time endowment, ε_t^H is idiosyncratic shock, and $\sum_{j=1}^{J} g_t^j(\eta_t^A) f^j\left(N_t^j(\varepsilon_0.,\varepsilon_t\,), L_t^j(\varepsilon_0.,\varepsilon_t\,) \right)$ is aggregate output of the economy.

The Lagrangian function of the above problem can be written as

$$\sum_{j=1}^{J} \omega^j \left[\sum_{\tau=0}^{T} \beta^\tau \sum_{(\varepsilon_0..\varepsilon_\tau)} \pi\,(\varepsilon_0.,\varepsilon_\tau) U^j \left[c_\tau^j(\varepsilon_0.,\varepsilon_\tau), \mathcal{L}_\tau^j(\varepsilon_0.,\varepsilon_\tau) \right] \right]$$

$$+ \sum_{\tau=0}^{T} \lambda_\tau(\varepsilon_0.,\varepsilon_\tau) \left[P_{q,\tau}(\eta_\tau^q) \sum_{j=1}^{J} Q_\tau^j(\varepsilon_0.,\varepsilon_\tau) + P_{a,\tau}(\eta_\tau^a) \left[\sum_{j=1}^{J} \left(\bar{\mathcal{L}}_\tau^j(\varepsilon_\tau^H) - \mathcal{L}_\tau^j(\varepsilon_0.,\varepsilon_\tau) \right) \right] \right]$$

$$- \sum_{\tau=0}^{T} \lambda_\tau(\varepsilon_0.,\varepsilon_\tau) \left[P_{q,\tau}(\eta_\tau^q) \sum_{j=1}^{J} c_\tau^j(\varepsilon_0.,\varepsilon_\tau) + P_{a,\tau}(\eta_\tau^a) \sum_{j=1}^{J} N_\tau^j(\varepsilon_0.,\varepsilon_\tau) + P_{r,\tau}(\eta_\tau^r) \sum_{j=1}^{J} L_\tau^j(\varepsilon_0.,\varepsilon_\tau) \right] \tag{5}$$

where $\lambda_t(\varepsilon_0.,\varepsilon_t)$ is a history, $(\varepsilon_0.,\varepsilon_t)$, contingent Lagrange multipliers at date t.

All the relevant first-order conditions to this problem, assuming no binding corner constraints for consumption $c_t^j \geq 0$, and leisure, $0 \leq \ell_t^j \leq \bar{\ell}_t^j$ are as follows,

$$\omega^j \beta^t \pi(\varepsilon_0,..,\varepsilon_t) U_{c_t^j}^j \left[c_t^j(\varepsilon_0,..,\varepsilon_t), \ell_t^j(\varepsilon_0,..,\varepsilon_t) \right] = \lambda_t(\varepsilon_0,..,\varepsilon_t) P_{q,t}(\eta_t^q),$$

$$t = 0,1,...,T ; j = 1,2,..., J.$$

(6)

$$\omega^j \beta^t \pi(\varepsilon_0,..,\varepsilon_t) U_{\ell_t^j}^j \left[c_t^j(\varepsilon_0,..,\varepsilon_t), \ell_t^j(\varepsilon_0,..,\varepsilon_t) \right] = \lambda_t(\varepsilon_0,..,\varepsilon_t) P_{a,t}(\eta_t^a),$$

$$t = 0,1,...,T ; j = 1,2,..., J.$$

(7)

$$\partial \left[g_t^j(\eta_t^A) f^j\left(N_t^j(\varepsilon_0,..,\varepsilon_t), L_t^j(\varepsilon_0,..,\varepsilon_t) \right) \right] \bigg/ \partial N_t^j(\varepsilon_0,...,\varepsilon_t)$$

(8)

$$= \frac{\lambda_t(\varepsilon_0,..,\varepsilon_t) P_{a,t}(\eta_t^a)}{\lambda_t(\varepsilon_0,..,\varepsilon_t) P_{q,t}(\eta_t^q)} = \frac{P_{a,t}(\eta_t^a)}{P_{q,t}(\eta_t^q)}, \quad t = 0,1,...,T ; j = 1,2,..., J.$$

$$\partial \left[g_t^j(\eta_t^A) f^j\left(N_t^j(\varepsilon_0,..,\varepsilon_t), L_t^j(\varepsilon_0,..,\varepsilon_t) \right) \right] \bigg/ \partial L_t^j(\varepsilon_0,...,\varepsilon_t)$$

(9)

$$= \frac{\lambda_t(\varepsilon_0,..,\varepsilon_t) P_{r,t}(\eta_t^r)}{\lambda_t(\varepsilon_0,..,\varepsilon_t) P_{q,t}(\eta_t^q)} = \frac{P_{r,t}(\eta_t^r)}{P_{q,t}(\eta_t^q)}, \quad t = 0,1,...,T ; j = 1,2,..., J.$$

Equation (6) and (7) state that, at each date t and state $(\varepsilon_0,...,\varepsilon_t)$, the present value of weighted expected marginal utility across households for consumption and leisure are equated across households to a common marginal utility of income. Equation (8) and (9) are conditions for production efficiency. They state that the marginal product of each factor of any household (j) must be equated to the price ratio of each factor and output. They hold over all households at date t and state $(\varepsilon_0,...,\varepsilon_t)$ so that production efficiency is obtained independent of household (j)'s leisure time endowment and its weight, ω^j. In summary, equilibrium conditions mean that

$$\omega^j \beta^t \pi(\varepsilon_0,..,\varepsilon_t) U_{c_t^j}^j \left[c_t^j(\varepsilon_0,..,\varepsilon_t), \ell_t^j(\varepsilon_0,..,\varepsilon_t) \right]$$

$$=...= \omega^J \beta^t \pi(\varepsilon_0,..,\varepsilon_t) U_{c_t^J}^J \left[c_t^J(\varepsilon_0,..,\varepsilon_t), \ell_t^J(\varepsilon_0,..,\varepsilon_t) \right],$$

(10)

$$\text{for all } j; j = 1,2,...,J., t = 0,1,2,...,T.$$

$$\partial\left[g_t^j(\eta_t^A)f^j\left(N_t^j(\varepsilon_{0,.},\varepsilon_t),\ L_t^j(\varepsilon_{0,.},\varepsilon_t)\right)\right]\bigg/\partial N_t^j(\varepsilon_{0,.},\varepsilon_t)$$

$$=\ldots=\partial\left[g_t^J(\eta_t^A)f^J\left(N_t^J(\varepsilon_{0,.},\varepsilon_t),\ L_t^J(\varepsilon_{0,.},\varepsilon_t)\right)\right]\bigg/\partial N_t^J(\varepsilon_{0,.},\varepsilon_t),$$ (11)

$$for\ all\ j;\ j=1,2,\ldots,J.,t=0,1,2,\ldots,T.$$

$$\partial\left[g_t^j(\eta_t^A)f^j\left(N_t^j(\varepsilon_{0,.},\varepsilon_t),\ L_t^j(\varepsilon_{0,.},\varepsilon_t)\right)\right]\bigg/\partial L_t^j(\varepsilon_{0,.},\varepsilon_t)$$

$$=\ldots=\partial\left[g_t^J(\eta_t^A)f^J\left(N_t^J(\varepsilon_{0,.},\varepsilon_t),\ L_t^J(\varepsilon_{0,.},\varepsilon_t)\right)\right]\bigg/\partial L_t^J(\varepsilon_{0,.},\varepsilon_t),$$ (12)

$$for\ all\ j;\ j=1,2,\ldots,J.,t=0,1,2,\ldots,T.$$

3.2 Total factor productivity in the full information economy

It can be shown that total factor productivity introduced by Solow (1957) can be different from average growth rate of technical change that one could get from a multi-period, multi-agent and state contingent model. If each household (j) has different linearly homogeneous production function as defined earlier in section 2, the economy has the following aggregate output.

$$\sum_{j=1}^{J}Q_t^j=\sum_{j=1}^{J}g_t^jf^j\left(N_t^j,L_t^j\right)$$ (13)

where $Q_t^j=Q_t^j(\varepsilon_{0,.},\varepsilon_t), g_t^j=g_t^j(\eta_t^A), N_t^j=N_t^j(\varepsilon_{0,.},\varepsilon_t),$ and $L_t^j=L_t^j(\varepsilon_{0,.},\varepsilon_t).$

Differentiating equation (13) totally with respect to time and one obtains

$$\sum_{j=1}^{J}\dot{Q}_t^j=\sum_{j=1}^{J}\dot{g}_t^jf^j\left(N_t^j,L_t^j\right)+\sum_{j=1}^{J}g_t^j\frac{\partial f^j}{\partial N_t^j}\dot{N}_t^j+\sum_{j=1}^{J}g_t^j\frac{\partial f^j}{\partial L_t^j}\dot{L}_t^j,$$ (14)

where dots indicate time derivative. By recalling those results from equation (8) and (9), one has

$$\frac{\partial Q_t^j}{\partial N_t^j}=g_t^j\frac{\partial f^j}{\partial N_t^j}=\frac{P_{a,t}(\eta_t^a)}{P_{q,t}(\eta_t^q)},$$ (15)

and $\quad \dfrac{\partial Q_t^j}{\partial L_t^j} = g_t^j \dfrac{\partial f^j}{\partial L_t^j} = \dfrac{P_{r,t}(\eta_t^r)}{P_{q,t}(\eta_t^q)},$ (16)

for all j = 1, 2, 3,..., J. In the equilibrium, equation (14) can then be rearranged as,

$$\sum_{j=1}^{J} \dot{Q}_t^j = \sum_{j=1}^{J} \dot{g}_t^j f^j\left(N_t^j, L_t^j\right) + \left[\dfrac{P_{a,t}(\eta_t^a)}{P_{q,t}(\eta_t^q)}\right] \sum_{j=1}^{J} \dot{N}_t^j + \left[\dfrac{P_{r,t}(\eta_t^r)}{P_{q,t}(\eta_t^q)}\right] \sum_{j=1}^{J} \dot{L}_t^j \quad (17)$$

Equation (17) can be rearranged into

$$\sum_{j=1}^{J} \dot{Q}_t^j = \sum_{j=1}^{J} \dot{g}_t^j f^j\left(N_t^j, L_t^j\right) + \left(w_{N,t}\right)\left[\sum_{j=1}^{J} Q_t^j\right]\left[\dfrac{\sum_{j=1}^{J} \dot{N}_t^j}{\sum_{j=1}^{J} N_t^j}\right] + \left(w_{L,t}\right)\left[\sum_{j=1}^{J} Q_t^j\right]\left[\dfrac{\sum_{j=1}^{J} \dot{L}_t^j}{\sum_{j=1}^{J} L_t^j}\right],$$

(18)

where $\quad w_{N,t} = w_{N,t}\left(\varepsilon_0,...,\varepsilon_t\right) = \left[\dfrac{P_{a,t}(\eta_t^a)}{P_{q,t}(\eta_t^q)}\right]\left[\dfrac{\sum_{j=1}^{J} N_t^j\left(\varepsilon_0,...,\varepsilon_t\right)}{\sum_{j=1}^{J} Q_t^j\left(\varepsilon_0,...,\varepsilon_t\right)}\right],$ (19)

and $\quad w_{L,t} = w_{L,t}\left(\varepsilon_0,...,\varepsilon_t\right) = \left[\dfrac{P_{r,t}(\eta_t^r)}{P_{q,t}(\eta_t^q)}\right]\left[\dfrac{\sum_{j=1}^{J} L_t^j\left(\varepsilon_0,...,\varepsilon_t\right)}{\sum_{j=1}^{J} Q_t^j\left(\varepsilon_0,...,\varepsilon_t\right)}\right]$ (20)

Then, multiply equation (18) by $\left(\sum_{j=1}^{J} Q_t^j\right)^{-1}$, one has the following growth rate of aggregate output

$$\dfrac{\sum_{j=1}^{J} \dot{Q}_t^j}{\sum_{j=1}^{J} Q_t^j} = \sum_{j=1}^{J}\left[\dfrac{Q_t^j}{\sum_{j=1}^{J} Q_t^j}\right]\left[\dfrac{\dot{g}_t^j}{g_t^j}\right] + \left(w_{N,t}\right)\left[\dfrac{\sum_{j=1}^{J} \dot{N}_t^j}{\sum_{j=1}^{J} N_t^j}\right] + \left(w_{L,t}\right)\left[\dfrac{\sum_{j=1}^{J} \dot{L}_t^j}{\sum_{j=1}^{J} L_t^j}\right] \quad (21)$$

The average growth rate of technical change weighted by each household's output share can be defined as

$$\sum_{j=1}^{J} \left[\frac{Q_t^j}{\sum_{j=1}^{J} Q_t^j} \right] \left[\frac{\dot{g}_t^j}{g_t^j} \right] = \frac{\sum_{j=1}^{J} \dot{Q}_t^j}{\sum_{j=1}^{J} Q_t^j} - (w_{N,t}) \left[\frac{\sum_{j=1}^{J} \dot{N}_t^j}{\sum_{j=1}^{J} N_t^j} \right] - (w_{L,t}) \left[\frac{\sum_{j=1}^{J} \dot{L}_t^j}{\sum_{j=1}^{J} L_t^j} \right] \qquad (22)$$

Equation (22) states that an average growth rate of technical change weighted by output share is equal to total output growth less output growth that was generated from the accumulation of labor and land inputs. This average growth rate of technical change, as displayed by the term on the left-hand side of equation (22), appears to include both the terms of Q_t^j and $\sum_{j=1}^{J} Q_t^j$. This is different from the concept of total factor productivity introduced by Solow (1957). Total factor productivity is referred to a source of output growth that arises from technical change and it is clearly separated from a source of output growth that arises from the accumulation of inputs. Solow was able to segregate variations in output growth due to technical change from those due to change in inputs by assuming for a linearly homogeneous aggregate production function. Solow's total factor productivity is then defined as

$$\frac{\dot{g}_t}{g_t} = \frac{\dot{Q}_t}{Q_t} - \overline{w}_{N,t} \frac{\dot{N}_t}{N_t} - \overline{w}_{L,t} \frac{\dot{L}_t}{L_t} \qquad (23)$$

where $Q_t = Q_t(\varepsilon_0,...,\varepsilon_t) = g_t(\eta_t^A) f\left(N_t(\varepsilon_0,...,\varepsilon_t), L_t(\varepsilon_0,...,\varepsilon_t)\right)$ is linearly homogeneous aggregate production function,

$N_t(\varepsilon_0,...,\varepsilon_t)$ is aggregate labor input, $L_t(\varepsilon_0,...,\varepsilon_t)$ is aggregate land input,

$$\overline{w}_{N,t} = \overline{w}_{N,t}(\varepsilon_0,...,\varepsilon_t) = \left[\frac{P_{a,t}(\eta_t^a)}{P_{q,t}(\eta_t^q)} \right] \left[\frac{N_t(\varepsilon_0,...,\varepsilon_t)}{Q_t(\varepsilon_0,...,\varepsilon_t)} \right], \text{ and}$$

$$\overline{w}_{L,t} = \overline{w}_{L,t}(\varepsilon_0,...,\varepsilon_t) = \left[\frac{P_{r,t}(\eta_t^r)}{P_{q,t}(\eta_t^q)} \right] \left[\frac{L_t(\varepsilon_0,...,\varepsilon_t)}{Q_t(\varepsilon_0,...,\varepsilon_t)} \right].$$

It can be shown that the result of adding up all J different linearly homogeneous production functions, denoted by $\sum\limits_{j=1}^{J} Q_t^j$, is not a linearly homogeneous aggregate production function.[3] In addition, let's assume for the following conditions,[4]

(a) The cumulated effect of shifts over time of a linearly homogeneous aggregate production function compared to those shift parameters from all households' linearly homogeneous production functions is such that

$$\min \left\{ g_t^1(\eta_t^A),..., g_t^J(\eta_t^A) \right\} \leq g_t(\eta_t^A) \leq \max \left\{ g_t^1(\eta_t^A),..., g_t^J(\eta_t^A) \right\};$$

(b) The labor share of the linearly homogeneous aggregate production function compared to those labor shares from all households'-linearly homogeneous production functions is such that

$$\min \left\{ \alpha_1,..., \alpha_J \right\} \leq \gamma \leq \max \left\{ \alpha_1,..., \alpha_J \right\},$$

where $0 < \alpha_j < 1$, for all j =1, 2,..., J., is household (j)'s labor share. Then one can conclude that, in the equilibrium, total output of a linearly homogeneous aggregate production function may be larger than, smaller than, or the same as total output of adding up all J different linearly homogeneous production functions. As a result, labor and land shares in equation (22) may be larger than, smaller than, or the same as those labor and land shares in equation (23), other things being equal.

If all households have identical linearly homogeneous production function, then equation (22) becomes the following equation.

$$\frac{\hat{\dot{g}}_t}{\hat{g}_t} = \frac{\hat{\dot{Q}}_t}{\hat{Q}_t} - w_{N,t} \frac{\hat{\dot{N}}_t}{\hat{N}_t} - w_{L,t} \frac{\hat{\dot{L}}_t}{\hat{L}_t} \qquad (24)$$

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[3] See the proof in appendix.

[4] These conditions are required to assure that the linearly homogeneous aggregate production function in equation (23) is also covered a special case of equation (22) where all households have identical linearly homogeneous production function.

where $\hat{Q}_t = Q_t^j(\varepsilon_0,...,\varepsilon_t)$, $\hat{N}_t = N_t^j(\varepsilon_0,...,\varepsilon_t)$, $\hat{L}_t = L_t^j(\varepsilon_0,...,\varepsilon_t)$, *for all j = 1,2,...,J,* $w_{Nt} = w_{Nt}(\varepsilon_0,...,\varepsilon_t)$, and $w_{Lt} = w_{Lt}(\varepsilon_0,...,\varepsilon_t)$. Note that equation (24) is also consistent with equation (23) given that both conditions above are satisfied.[5] The growth rate of technical change, as displayed by the term on the left-hand side of equation (24), is related neither to the accumulation of labor nor to the accumulation of land. This is true because total output from adding up all J identical linearly homogeneous production functions, or, $J. Q_t$, is also a linearly homogeneous aggregate production function. In this case, each household will always have the same value of total factor productivity.

Therefore, one can only get the same total factor productivity as proposed by Solow (1957) from these two cases: (a) there is a linearly homogeneous aggregate production function; or (b) each household has an identical linearly homogeneous production function.

It is interesting to ask whether these findings provide new additional explanations to the results found by some empirical studies on total factor productivity in Thailand. Akrasanee and Wiboonchutikula (1994) found that the rate of total factor productivity growth of Thai manufacturing was low compared to many developed countries during 1963-1986. In addition, despite the increase in real inputs declined over time relative to total factor productivity growth, the increase of total factor productivity contribution was small. They also found that total factor productivity growths of Thai manufacturing had quite large variations from time to time, for example, total factor productivity growths for the periods of 1963-72, 1972-74, 1974-79, 1979-82 and 1982-86 were 0.31, -0.78, 2.40, -1.42 and 2.25, respectively. Tinakorn and Sussangkarn (1994) found that

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[5] This point can be easily proved by setting $g_t(\eta_t^A) = \hat{g}_t(\eta_t^A)$ and $\gamma = \alpha$; where α is the labor share of any household (j). Note that both conditions also fulfill condition (a) and (b) mentioned earlier. Then one can have $Q_t = J. \hat{Q}_t$, given that both (22) and (23) have the same numbers of aggregate labor and land inputs.

during 1981-1990, the contribution of total factor productivity to the total growth of output in the non-agricultural sectors was small. They found that the contribution to output of capital in these sectors was as high as 40-65 percent. In addition, total factor productivity in agriculture accounted for 25 percent of the total growth in output during the same period. They also found that the variations of total factor productivity growths of agricultural sector were less than those variations of non-agricultural sectors. For example, for the periods of 1978-1981, 1982-1986 and 1987-1990, total factor productivity growths of agricultural sector, adjusted for input quality, were 0.48, 1.40 and 1.97, respectively. Adjusted total factor productivity growths of industry were -2.93, -1.99 and 3.42, respectively, while adjusted total factor productivity growths of manufacturing were -2.81, 0.14 and 1.48, respectively. Note that during these periods, all three sectors' output growths were on the rise as indicated by Figure 1, 2 and 3. By comparing the indices of output and inputs in agricultural sector as shown by Figure 1, it was clearly seen that agricultural output growths were higher than all input growths for many years, hence resulted in positive values of total factor productivity growths. In addition, both Figure 2 and 3 shown that capital and labor growths were higher than output growths in the early years, hence resulted in negative values of total factor productivity growths of industry and manufacturing. Output growths of these sectors were, later, higher than labor growths and also moved closer to capital growths. Hence, it resulted in positive values of total factor productivity growths in these sectors in later periods. All these interpretations are fitting well with the fact that factor income shares of each sector during these periods were not fluctuated very much as shown by Figure 4, 5 and 6.

In general, aggregate outputs or gross domestic products are measured by using data from some base year surveys to account for outputs from different production units in the economy.[6] Hence, total factor

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[6] See more details in Department of International Economic and Social Affairs, Statistical Office (1986).

productivity growths from both studies mentioned above are better described by the term on the left-hand side of equation (22) than by the term on the left-hand side of equation (23). When households face different linearly homogenous production functions, an average growth rate of technical change could be different from Solow's total factor productivity because the estimated average growth of technical change is weighted by output share. Its implication is that total factor productivity growths of some sectors could be highly sensitive to output of a few large enterprises with huge output shares and high capital intensity. It may also help to explain the negative values of total factor productivity growths in the early period before they turned positive as shown in Tinakorn and Sussangkarn (1994), since output growths of new large enterprises tend to follow capital growths due to time lags. It can be concluded that, under such circumstance, one can get better estimates of total factor productivity growths of industry and manufacturing if one chooses to estimate total factor productivity growths of large and small firms separately.

3.3 The case of rent sharing[7]

In the full information economy,[8] the risk-neutral landlord is able to fully observe each household (j)'s real labor effort at any date. Optimal decisions can be obtained by solving,

$$
\underset{l_t^j, c_t^j, N_t^j, L_t^j}{Max} \sum_{j=1}^{J} \omega^j \left[\sum_{t=0}^{T} \beta^t \sum_{(\varepsilon_0 \cdots \varepsilon_t)} \pi(\varepsilon_0 \cdots \varepsilon_t) U^j \left[c_t^j(\varepsilon_0 \cdots \varepsilon_t), l_t^j(\varepsilon_0 \cdots \varepsilon_t) \right] \right] +
$$

$$
\omega^{J+1} \left[\sum_{t=0}^{T} \beta^t \sum_{(\varepsilon_0 \cdots \varepsilon_t)} \pi(\varepsilon_0 \cdots \varepsilon_t) \left[\sum_{j=1}^{J} Q_t^j(\varepsilon_0 \cdots \varepsilon_t) - \sum_{j=1}^{J} c_t^j(\varepsilon_0 \cdots \varepsilon_t) \right] \right] \quad (25)
$$

subject to (a) budget constraint,

---

[7] The statistics from the Office of Agricultural Economics in 1998 shown that all types of rental farms in Thailand were 14,781,910 rais or 11.33 percent of total farming land. The number of rent-sharing agreement varies considerably from crop to crop.

[8] Rent-sharing agreement under imperfect information should be a better model to represent problems in the real world. The topic will be explored as another separate paper in the future.

$$P_{q,t}(\eta_t^q)\sum_{j=1}^{J}c_t^j(\varepsilon_0,...,\varepsilon_t)+P_{a,t}(\eta_t^a)\sum_{j=1}^{J}N_t^j(\varepsilon_0,...,\varepsilon_t)+P_{r,t}^j(\varepsilon_0,...,\varepsilon_t)$$

$$=P_{q,t}(\eta_t^q)\sum_{j=1}^{J}g_t^j(\eta_t^A)f^j\left(N_t^j(\varepsilon_0,...,\varepsilon_t),L_t^j(\varepsilon_0,...,\varepsilon_t)\right)+P_{a,t}(\eta_t^a)\left[\sum_{j=1}^{J}\left(\bar{\ell}_t^j(\varepsilon_t^H)-\ell_t^j(\varepsilon_0,...,\varepsilon_t)\right)\right],$$

$$t = 0,1,...,T \; ; \; j = 1,2,..., J.$$

(26)

where

$$P_{r,t}^j(\varepsilon_0,...,\varepsilon_t) = P_{q,t}(\eta_t^q)\cdot\left[Q_t^j(\varepsilon_0,...,\varepsilon_t)-c_t^j(\varepsilon_0,...,\varepsilon_t)\right],$$ (27)

$$t = 0,1,...,T \; ; \; j = 1,2,..., J.$$

and (b) land constraint,

$$\sum_{j=1}^{J}L_t^j(\varepsilon_0,...,\varepsilon_t)=\bar{L},$$ (28)

where $\bar{L}$ is total land input.

Equation (26) simply says that household (j)'s total expenditure, which included consumption expense, wage payment and part of his output values paid to landlord for land rent, must be equal to his total income. Note that equation (26) can be rearranged into,

$$P_{a,t}(\eta_t^a)\sum_{j=1}^{J}N_t^j(\varepsilon_0,...,\varepsilon_t)=P_{a,t}(\eta_t^a)\left[\sum_{j=1}^{J}\left(\bar{\ell}_t^j(\varepsilon_t^H)-\ell_t^j(\varepsilon_0,...,\varepsilon_t)\right)\right],$$ (29)

$$t = 0,1,...,T \; ; \; j = 1,2,..., J.$$

where consumption of the landlord at any time t and state $(\varepsilon_0,...,\varepsilon_t)$ is denoted by,

$$c_t^{J+1}(\varepsilon_0,...,\varepsilon_t)=\sum_{j=1}^{J}g_t^j(\eta_t^A)f^j\left(N_t^j(\varepsilon_0,...,\varepsilon_t),L_t^j(\varepsilon_0,...,\varepsilon_t)\right)-\sum_{j=1}^{J}c_t^j(\varepsilon_0,...,\varepsilon_t)$$ (30)

The first-order conditions of this problem are,

$$\omega^j\beta^t\pi(\varepsilon_0,...,\varepsilon_t)U_{c_t^j}^j\left[c_t^j(\varepsilon_0,...,\varepsilon_t),\ell_t^j(\varepsilon_0,...,\varepsilon_t)\right]=\omega^{J+1}\beta^t\pi(\varepsilon_0,...,\varepsilon_t),$$ (31)

$$t = 0,1,...,T \; ; \; j = 1,2,..., J.$$

$$\omega^j \beta^t \pi(\varepsilon_0,..,\varepsilon_t) U_{\ell,t}^j \left[c_t^j(\varepsilon_0,..,\varepsilon_t), \ell_t^j(\varepsilon_0,..,\varepsilon_t) \right] = \lambda_t(\varepsilon_0,..,\varepsilon_t) P_{a,t}(\eta_t^a), \qquad (32)$$

$$t = 0,1,...,T \ ; \ j = 1,2,..., J.$$

$$\partial \left[g_t^j(\eta_t^A) f^j \left(N_t^j(\varepsilon_0,..,\varepsilon_t), L_t^j(\varepsilon_0,..,\varepsilon_t) \right) \right] \Big/ \partial N_t^j(\varepsilon_0,..,\varepsilon_t) = \frac{\lambda_t(\varepsilon_0,..,\varepsilon_t) P_{a,t}(\eta_t^a)}{\omega^{J+1} \beta^t \pi(\varepsilon_0,..,\varepsilon_t)}. \qquad (33)$$

$$t = 0,1,...,T \ ; \ j = 1,2,..., J.$$

$$\partial \left[g_t^j(\eta_t^A) f^j \left(N_t^j(\varepsilon_0,..,\varepsilon_t), L_t^j(\varepsilon_0,..,\varepsilon_t) \right) \right] \Big/ \partial L_t^j(\varepsilon_0,..,\varepsilon_t) = \frac{\mu_t(\varepsilon_0,..,\varepsilon_t)}{\omega^{J+1} \beta^t \pi(\varepsilon_0,..,\varepsilon_t)}. \qquad (34)$$

$$t = 0,1,...,T \ ; \ j = 1,2,..., J.$$

where $\lambda_t(\varepsilon_0,...,\varepsilon_t)$ and $\mu_t(\varepsilon_0,...,\varepsilon_t)$ are history contingent Lagrange multipliers at date t of constraint (29) and (28), respectively.

Equation (33) states that, in equilibrium, all households have the same marginal product of labor. Similarly, equation (34) states that all households have the same marginal product of land. By using all the above conditions, one can find the optimal amount of consumption, labor effort and land input of each household (j) at any date t and state $(\varepsilon_0,..,\varepsilon_t)$.[9]

In the case of rent sharing agreement and all households have different linearly homogeneous production functions, total factor productivity growth as analogous to equation (22) is

$$\sum_{j=1}^{J} \left[\frac{\tilde{Q}_t^j}{\sum_{j=1}^{J} \tilde{Q}_t^j} \right] \left[\frac{\dot{\tilde{g}}_t^j}{\tilde{g}_t^j} \right] = \frac{\sum_{j=1}^{J} \dot{\tilde{Q}}_t^j}{\sum_{j=1}^{J} \tilde{Q}_t^j} - (\tilde{w}_{N,t}) \left[\frac{\sum_{j=1}^{J} \dot{\tilde{N}}_t^j}{\sum_{j=1}^{J} \tilde{N}_t^j} \right] - (\tilde{w}_{L,t}) \left[\frac{\sum_{j=1}^{J} \dot{\tilde{L}}_t^j}{\sum_{j=1}^{J} \tilde{L}_t^j} \right] \qquad (35)$$

---

[9] If the landlord is risk averse, then the right-hand side of equation (33) becomes ,

$$\frac{\lambda_t(\varepsilon_0,...,\varepsilon_t) P_{a,t}(\eta_t^a)}{\omega^{J+1} \beta^t \pi(\varepsilon_0,..,\varepsilon_t) U_c^{J+1}},$$ where $U_c^{J+1} > 0$ is marginal utility for consumption of the landlord.

This implies that, in the equilibrium, all households still have the same marginal product of labor. By the same argument, similar result can be applied to marginal product of land shown in equation (34).

where

$$\tilde{w}_{N,t} = \tilde{w}_{N,t}(\varepsilon_0,...,\varepsilon_t) = \tag{36}$$

$$\left[\frac{\lambda_t(\varepsilon_0,...,\varepsilon_t)P_{a,t}(\eta_t^a)}{\omega^{J+1}\beta^t\pi(\varepsilon_0,..,\varepsilon_t)}\right]\left[\frac{\sum_{j=1}^{J}\tilde{N}_t^j(\varepsilon_0,...,\varepsilon_t)}{\sum_{j=1}^{J}\tilde{Q}_t^j(\varepsilon_0,...,\varepsilon_t)}\right],$$

and

$$\tilde{w}_{L,t} = \tilde{w}_{L,t}(\varepsilon_0,...,\varepsilon_t) = \tag{37}$$

$$\left[\frac{\mu_t(\varepsilon_0,...,\varepsilon_t)}{\omega^{J+1}\beta^t\pi(\varepsilon_0,..,\varepsilon_t)}\right]\left[\frac{\sum_{j=1}^{J}\tilde{L}_t^j(\varepsilon_0,...,\varepsilon_t)}{\sum_{j=1}^{J}\tilde{Q}_t^j(\varepsilon_0,...,\varepsilon_t)}\right].$$

In the case of rent sharing, total factor productivity growth as shown by equation (35) is again different from Solow's total factor productivity growth. Hence, similar implications that found in section 3.2 are also applied to this case. Note that in this case, marginal products of labor and land are dependent to the landlord's attitude to risk as indicated by equation (33) and (34).

4. Conclusion

It was shown that under the full information with multi-period and multi-agent setting, average growth of technical change could be different from Solow's total factor productivity if households have different linearly homogenous production functions. Nevertheless, it is likely that the difference becomes smaller when the majority of households have the same form of production function. Similar results can also be obtained from the case of rent-sharing model. In the equilibrium, all households have the same marginal product of labor and land. However, both marginal products are dependent to the landlord's attitude to risk as indicated by equation (33) and (34).

In general, aggregate outputs or gross domestic products are measured by using data from some base year surveys to account for outputs from different production units in the economy. Hence, total factor productivity growths are better described by the term on the left-hand side of equation (22) than the term on the left-hand side of equation (23). When households face different linearly homogenous production functions, an average growth rate of technical change could be different from Solow's total factor productivity because the estimated average growth of technical change is weighted by output share. Its implication is that total factor productivity growths of some sectors could be affected by output of a few large enterprises with huge output shares and high capital intensity.

The implication for applied research is that one could get better estimates of total factor productivity growths for some sectors if one chooses to estimate total factor productivity growths of large and small firms separately.

Appendix

For simplicity, let's use the example of a two-household case for illustration. The first and second households have the following production functions, respectively.

$$Q_t^1 = g_t^1 \left(\eta_t^A \right) . \left(N_t^1 \right)^{(\alpha)} . \left(L_t^1 \right)^{(1-\alpha)} \tag{a.1}$$

and $$Q_t^2 = g_t^2 \left(\eta_t^A \right) . \left(N_t^2 \right)^{(\beta)} . \left(L_t^2 \right)^{(1-\beta)}, \tag{a.2}$$

where $g_t^1 \left(\eta_t^A \right) > 0$, $g_t^2 \left(\eta_t^A \right) > 0$, $g_t^1 \left(\eta_t^A \right) \neq g_t^2 \left(\eta_t^A \right)$, $0 < \alpha, \beta < 1$, $\alpha \neq \beta$.

The marginal product of labor of household 1 is

$$\frac{\partial Q_t^1}{\partial N_t^1} = \alpha . g_t^1 \left(\eta_t^A \right) . \left(N_t^1 \right)^{(\alpha-1)} . \left(L_t^1 \right)^{(1-\alpha)} \tag{a.3}$$

By recalling the result from equation (8), one can, in equilibrium, equate the marginal product of labor of the first household to the price ratio of labor and output. Hence,

$$\frac{\partial Q_t^1}{\partial N_t^1} = \alpha \cdot g_t^1\left(\eta_t^A\right) \cdot \left(N_t^1\right)^{(\alpha-1)}\left(L_t^1\right)^{(1-\alpha)} = \frac{P_{a,t}\left(\eta_t^a\right)}{P_{q,t}\left(\eta_t^q\right)} \tag{a.4}$$

By rearranging equation (a.4), one has

$$N_t^1 = \left[\frac{P_{a,t}\left(\eta_t^a\right)}{\alpha \cdot g_t^1\left(\eta_t^A\right) \cdot P_{q,t}\left(\eta_t^q\right)}\right]^{\left[\frac{1}{\alpha-1}\right]} L_t^1 \tag{a.5}$$

By substituting N_t^1 from equation (a.5) into equation (a.1), one can have

$$Q_t^1 = \left(g_t^1\left(\eta_t^A\right)\right)^{\left[\frac{1}{1-\alpha}\right]}\left[\frac{\alpha P_{q,t}\left(\eta_t^q\right)}{P_{a,t}\left(\eta_t^a\right)}\right]^{\left[\frac{\alpha}{1-\alpha}\right]} L_t^1 \tag{a.6}$$

By the same analogy, the production function of the second household can be written as

$$Q_t^2 = \left(g_t^2\left(\eta_t^A\right)\right)^{\left[\frac{1}{1-\beta}\right]}\left[\frac{\beta P_{q,t}\left(\eta_t^q\right)}{P_{a,t}\left(\eta_t^a\right)}\right]^{\left[\frac{\beta}{1-\beta}\right]} L_t^2 \tag{a.7}$$

By adding both households' output, one has

$$Q_t^1 + Q_t^2 = \left(g_t^1\left(\eta_t^A\right)\right)^{\left[\frac{1}{1-\alpha}\right]}\left[\frac{\alpha P_{q,t}\left(\eta_t^q\right)}{P_{a,t}\left(\eta_t^a\right)}\right]^{\left[\frac{\alpha}{1-\alpha}\right]} L_t^1 + \left(g_t^2\left(\eta_t^A\right)\right)^{\left[\frac{1}{1-\beta}\right]}\left[\frac{\beta P_{q,t}\left(\eta_t^q\right)}{P_{a,t}\left(\eta_t^a\right)}\right]^{\left[\frac{\beta}{1-\beta}\right]} L_t^2 \tag{a.8}$$

It can be seen that the outcome of equation (a.8) is not a linearly homogeneous production function. Hence, the result of adding up two different linearly homogeneous production functions is not a linearly

homogeneous aggregate production function.

Now, suppose that there is a linearly homogenous aggregate production for this economy. By giving the same amount of total labor and land inputs as the previous case, a relevant linearly homogeneous aggregate production function can be written as follow.

$$Q_t = g_t\left(\eta_t^A\right).\left(N_t^1 + N_t^2\right)^{(\gamma)}.\left(L_t^1 + L_t^2\right)^{(1-\gamma)} \tag{a.9}$$

where $g_t\left(\eta_t^A\right) > 0$ and $0 < \gamma < 1$.

Since, in equilibrium, the marginal product of labor from this aggregate production function must also be equated to the same price ratio of labor and output, one can surely obtain

$$Q_t = \left(g_t\left(\eta_t^A\right)\right)^{\left[\frac{1}{1-\gamma}\right]}\left[\frac{\gamma P_{q,t}\left(\eta_t^q\right)}{P_{a,t}\left(\eta_t^a\right)}\right]^{\left[\frac{\gamma}{1-\gamma}\right]}\left(L_t^1 + L_t^2\right) \tag{a.10}$$

Equation (a.10) can be rewritten as

$$Q_t = \left(g_t\left(\eta_t^A\right)\right)^{\left[\frac{1}{1-\gamma}\right]}\left[\frac{\gamma P_{q,t}\left(\eta_t^q\right)}{P_{a,t}\left(\eta_t^a\right)}\right]^{\left[\frac{\gamma}{1-\gamma}\right]}\left(L_t^1\right) + \left(g_t\left(\eta_t^A\right)\right)^{\left[\frac{1}{(1-\gamma)}\right]}\left[\frac{\gamma P_{q,t}\left(\eta_t^q\right)}{P_{a,t}\left(\eta_t^a\right)}\right]^{\left[\frac{\gamma}{1-\gamma}\right]}\left(L_t^2\right) \tag{a.11}$$

Giving that both cases have the same total amount of each type of input and the same price ratio of labor and output, total output from a linearly homogeneous aggregate production function can either be greater or less than total output from adding up two different linearly homogeneous production functions.

The following example will illustrate this point clearly. Suppose that

(a) The cumulated effect of shifts over time of the linearly homogeneous aggregate production function as compared to those shift parameters of both households' linearly homogeneous production functions is such that,

$$\min \left\{ g_t^1(\eta_t^A), \ g_t^2(\eta_t^A) \right\} \le g_t(\eta_t^A) \le \max \left\{ g_t^1(\eta_t^A), g_t^2(\eta_t^A) \right\};$$

(b) The labor share of the linearly homogeneous aggregate production function as compared to those labor shares of both households' linearly homogeneous production functions is such that,

$$\min \left\{ \alpha, \beta \right\} \le \gamma \le \max \left\{ \alpha, \beta \right\}.$$

Then let's subtract equation (a.8) out of equation (a.11). One can conclude that, in the equilibrium and if other things being equal, total output of a linearly homogeneous aggregate production function may be larger than, smaller than or the same as total output of adding up two different linearly homogeneous production functions. This result is clearly shown by the following equation that both terms on its right-hand side may have positive, negative, or zero value.

$$Q_t - \left(Q_t^1 + Q_t^2 \right) \qquad \text{(a.12)}$$

$$= \left\{ \left(g_t(\eta_t^A) \right)^{\left[\frac{1}{1 \cdot \gamma} \right]} \left[\frac{\gamma P_{q,t}(\eta_t^q)}{P_{a,t}(\eta_t^a)} \right]^{\left[\frac{\gamma}{1 \cdot \gamma} \right]} - \left(g_t^1(\eta_t^A) \right)^{\left[\frac{1}{1 \cdot \alpha} \right]} \left[\frac{\alpha P_{q,t}(\eta_t^q)}{P_{a,t}(\eta_t^a)} \right]^{\left[\frac{\alpha}{1 \cdot \alpha} \right]} \right\} \left(L_t^1 \right)$$

$$+ \left\{ \left(g_t(\eta_t^A) \right)^{\left[\frac{1}{1 \cdot \gamma} \right]} \left[\frac{\gamma P_{q,t}(\eta_t^q)}{P_{a,t}(\eta_t^a)} \right]^{\left[\frac{\gamma}{1 \cdot \gamma} \right]} - \left(g_t^2(\eta_t^A) \right)^{\left[\frac{1}{1 \cdot \beta} \right]} \left[\frac{\beta P_{q,t}(\eta_t^q)}{P_{a,t}(\eta_t^a)} \right]^{\left[\frac{\beta}{1 \cdot \beta} \right]} \right\} \left(L_t^2 \right)$$

$$\gtrless 0$$

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Figure 1. Indices of Output and Input in Agriculture
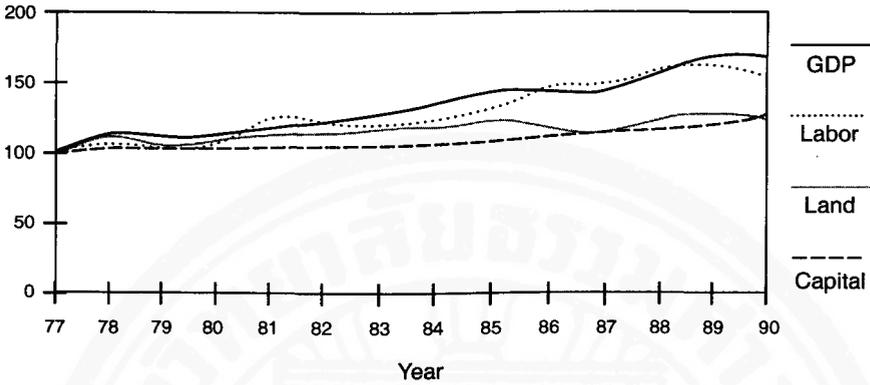
Figure 2. Indices of Output and Input in Industry
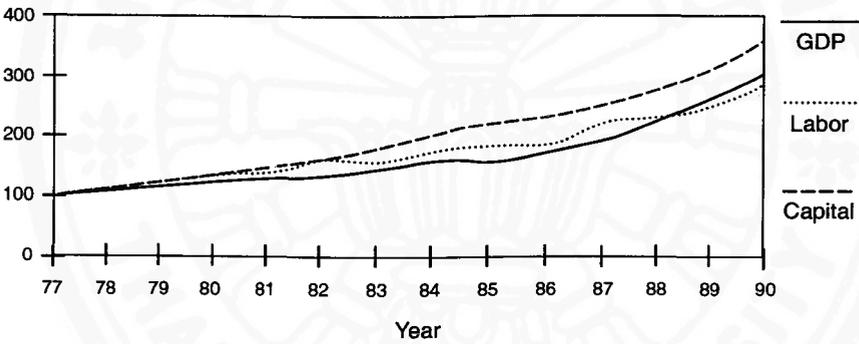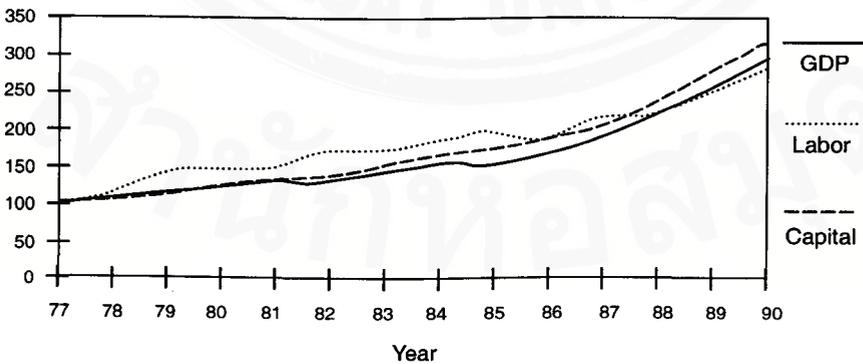

Figure 3. Indices of Output and Input in Manufacturing

Source: Data from Table 5.1-5.3 in Tinakorn and Sussangkarn (1994) were used to plot Figure 1-3.
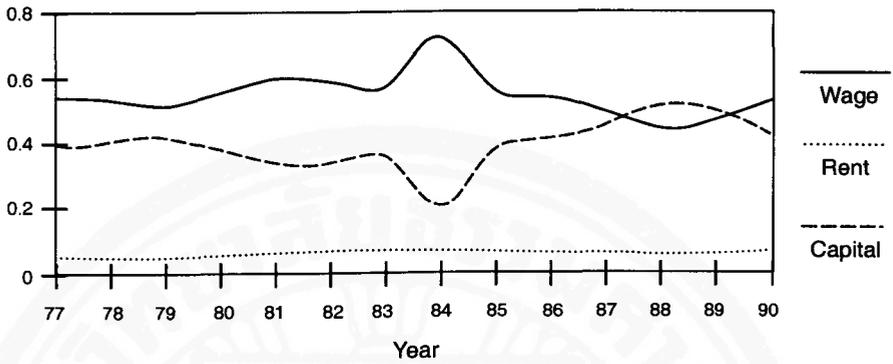
Figure 4. Factor Income Shares in Agriculture
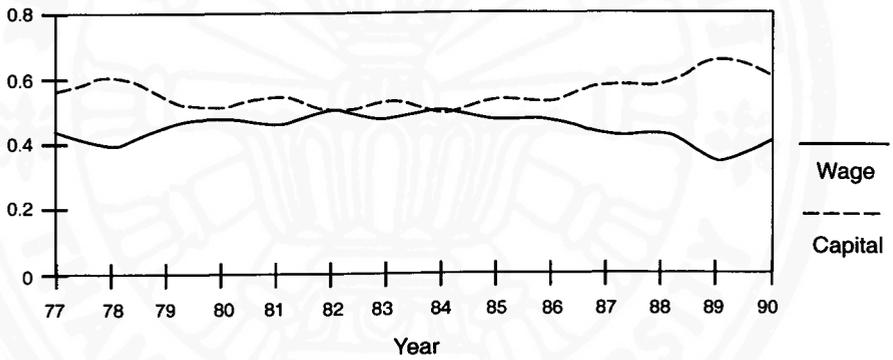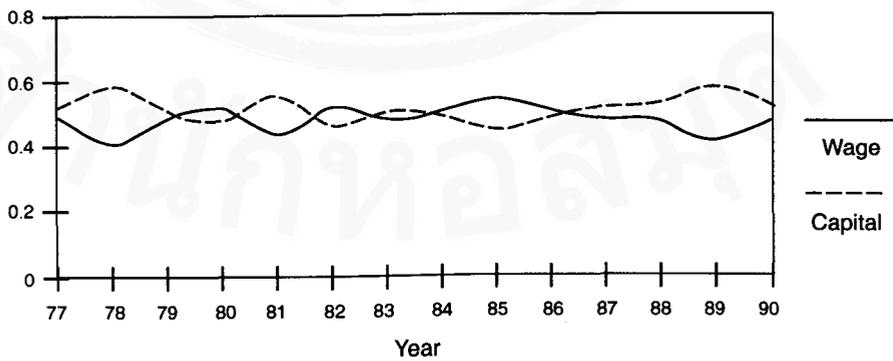

Figure 5. Factor Income Shares in Industry

Figure 6. Factor Income Shares in Manufacturing

Source: Data from Table 5.13-5.14 in Tinakron and Sussangkarn (1994) were used to plot

Figure 4-6.